COMPARISON OF THE MITTAG-LEFFLER FUNCTION AND LAGUERRE FUNCTIONS FOR EVALUATING THE INVERSE LAPLACE TRANSFORM

Vilem Karsky
Doctoral Degree Programme (2), FEEC BUT
E-mail: xkarsk010@stud.feec.vutbr.cz

Supervised by: Pavel Jura
E-mail: jura@feec.vutbr.cz

Abstract: This paper focuses on the evaluation inverse Laplace transform of the fractional order transfer functions. There are shown two methods how to compute inverse Laplace transform. First method uses Mittag-Leffler functions and the second method employs generalized Laguerre functions. These methods will be also compared.

Keywords: Mittag-Leffler functions, Generalized Laguerre functions, Fractional order transfer function, Inverse Laplace transform

1 INTRODUCTION

At this time it is possible to meet fractional order system more often. It is because this description is much accurate for some systems. You can meet them in quantum mechanics, voice and image processing, speech recognition and synthesis... The main drawback of this description is that all computations are much complex, mainly inverse Laplace transform. In this paper will be compared two methods for computation of the inverse Laplace transform. One method is based on Mittag-Leffler functions and second employs generalized Laguerre functions.

2 MATHEMATICAL BACKGROUND

2.1 GAMMA FUNCTION

Gamma function is generalization of factorial in form of integral. This function is defined

$$\Gamma(z) = \int_0^\infty x^{z-1} \mathrm{e}^{-x} \mathrm{d}x. \tag{1}$$

Relationship between Gamma function and factorial is described by this formula

$$\Gamma(n+1) = n! \ \ ; n \in \mathbb{N}. \tag{2}$$

2.2 MITTAG-LEFFLER FUNCTIONS

We can meet exponential functions in integer order calculus quite often. But in the fractional order calculus we meet Mittag-Leffler functions (MLF) instead. MLF are generalization of the exponential function in form of the infinite series. MLF are defined

$$E_{\alpha;\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad [1]. \tag{3}$$

MLF are identical with the exponential function for $\alpha = \beta = 1$.

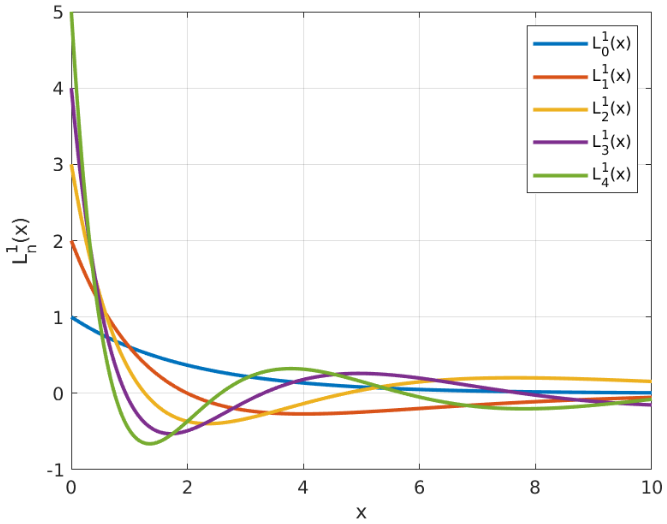

Figure 1: Generalized Laguerre functions

2.3 Generalized Laguerre functions

Generalized Laguerre functions we can define, according to [2], as

$$l_i^\alpha(t) = \sqrt{2\lambda}\mathrm{e}^{-\lambda t} L_i^\alpha(2\lambda t), \tag{4}$$

where $L_i^\alpha(t)$ are generalized Laguerre polynomials. Generalized Laguerre polynomials are defined as

$$L_n^\alpha(x) = \frac{\mathrm{e}^x x^{-\alpha}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+\alpha}\mathrm{e}^{-x}\right) \quad [3]. \tag{5}$$

GLF are shown in Figure 1.

2.4 Laplace transform of fractional order derivative

As written in [2], the Laplace transform of the Caputo derivative is

$$\mathscr{L}\left\{D_0^\alpha f(t)\right\}(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0). \tag{6}$$

Inverse Laplace transform of fractional order transfer function can be obtained by Podlubny's formula

$$\mathscr{L}\left\{t^{\alpha k+\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^\alpha)\right\} = \frac{k! s^{\alpha-\beta}}{(s^\alpha \mp a)^{k+1}} \quad [1], \tag{7}$$

or by employing Generalized Laguerre functions with $\alpha = 1$ as described in [2]. In second case we get solution in form

$$g(t) = \sum_{i=0}^{\infty} c_i^1 l_i^1 = \sqrt{2\lambda}\mathrm{e}^{-\lambda t}\sum_{i=0}^{\infty} c_i^1 L_i^1(2\lambda t), \tag{8}$$

where c_i^1 are spectrum coefficients of the transfer function in generalized Laguerre functions base.

3 COMPARISON OF BOTH METHODS

For comparison of both methods was chosen two transfer functions (first is integer order transfer function and sescond is fractional order transfer function). For almost all computations the authors Matlab toolbox [4] was used.

3.1 Integer order system

Firs system is defined by this transfer function

$$F(s) = \frac{10}{s^2 + 4s + 10}. \tag{9}$$

Analytical solution of this transfer function is

$$g_a(t) = 4.08\mathrm{e}^{-2t} \sin(2.45t). \tag{10}$$

This impulse response is plotted in Figure 2 with blue line. For this calculation was employed 40 terms of the MLF.

When we use formula (3) we get impulse response in form

$$g_m(t) = 2.04\mathrm{j}\left[E_{1;1}(-(2+2.45\mathrm{j})t) - E_{1;1}(-(2-2.45\mathrm{j})t)\right]. \tag{11}$$

This impulse response can be modified to equation (10), but it is possible only for integer order system. In Figure 2 the impulse response, which is obtained directly from toolbox, is drawn with orange line.

Third way to get impulse response is by using Generalized Laguerre functions as mentioned earlier. This impulse response will be in form (8). For this system was employed only first 7 generalized Laguerre functions with $\lambda = 4.4645$. In Table 1 you can see spectrum coefficients. This impulse response is plotted in Figure 2 with yellow line.

Table 1: The coefficients' spectrum: integer order system

i	0	1	2	3	4	5	6
c_i^1	1.5104	-1.0951	0.0735	0.2461	-0.1208	-0.0119	0.0356

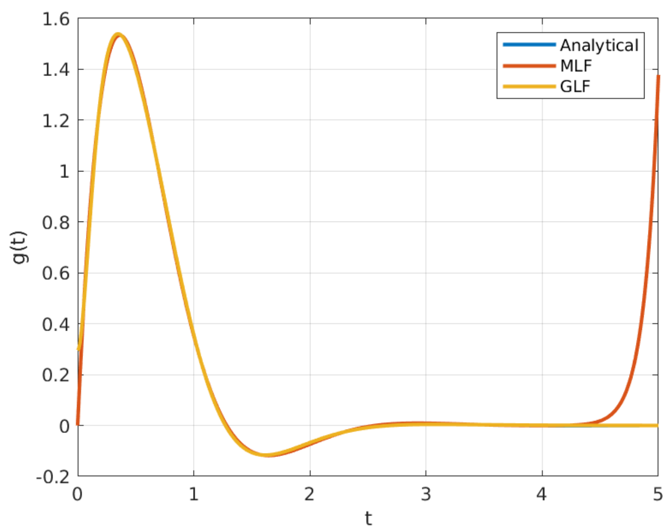

Figure 2: Impulse response of the first system

For comparison of the approximations was calculated two differences $g_a(t) - g_m(t)$ and $g_a(t) - g_g(t)$. These differences are shown in Figure 3. You can see that MLF better approximate impulse response in first part but, then they diverge. In opposite GLF have some approximation error in the beginning, and then they are converging to $g(t)$. It is worth mention that for MLF was used 40 terms and for GLF was used only 7 terms.

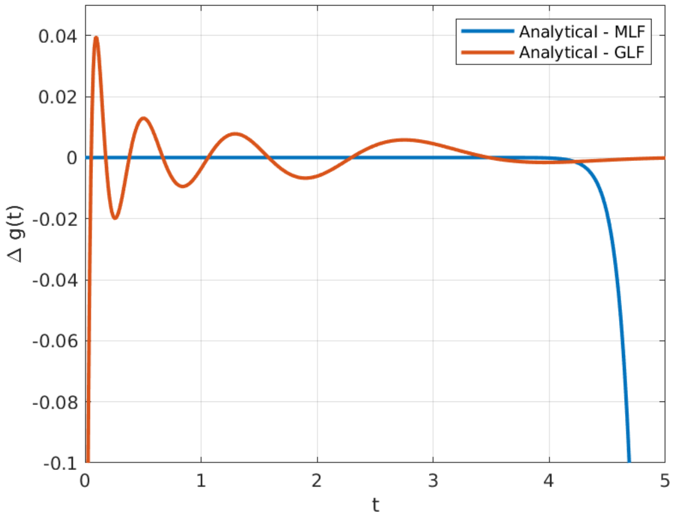

Figure 3: Differences from the $g_a(t)$

3.2 Fractional order system

For fractional order transfer function was chosen quite similar system. This system is described

$$F(s) = \frac{10}{s^{1.2} + 4s^{0.6} + 10}. \tag{12}$$

If we use formula (3) to get impulse response, we get

$$g_m(t) = 2.041\mathrm{j}t^{-0.4}\left[E_{0.6;0.6}(-(2+2.45j)t^{0.6} - E_{0.6;0.6}(-(2-2.45j)t^{0.6}\right]. \tag{13}$$

You can see it in Figure 4 with blue line. For computation was used first 200 terms.

But when we employ first 7 GLF with $\lambda = 5.7198$ we get spectrum coefficients, which are in Table 2. It's plotted in Figure 4 with orange line.

Table 2: The coefficients' spectrum: fractional order system

i	0	1	2	3	4	5	6
c_i^1	1.2877	-0.2196	0.1522	-0.0852	0.0576	-0.0453	0.0315

As you can see in Figure 4 both methods give similar result, but MLF diverge.

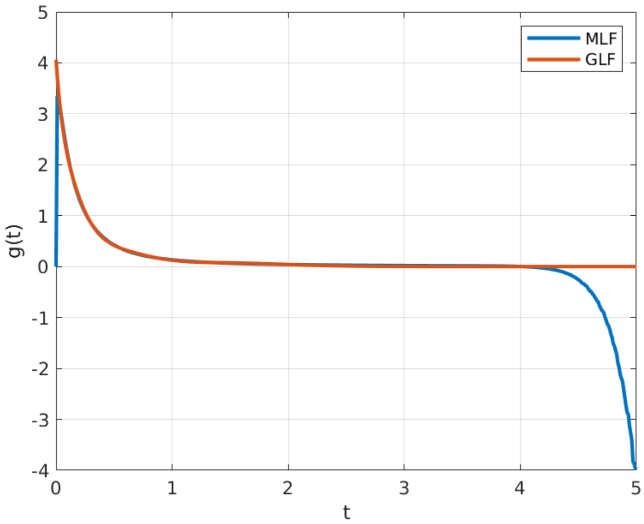

Figure 4: Impulse response of the second system

4 CONCLUSION

In this paper was shown that both methods are suitable for approximation impulse response. Solution obtained with MLF offers better results in the beginning of the impulse response but it diverges and needs a lot of terms. On the other hand solution using GLF converges and needs only a few terms but the approximation of the beginning of the impulse response is little worse.

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